On the Large Selberg Class

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Introduction

Beyond the Selberg class

The basic results

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All of these are examples of what are known as unitary automorphic L-functions.

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(P3) There exist a real number $Q > 0$, a complex number ω such that $|\omega| = 1$, and a function $G(s)$ of the form

$$G(s) = \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j), \tag{1}$$

where $\lambda_j > 0$ and $\mu_j \in \mathbb{C}$, such that

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(P4) The function $F(s)$ can be expressed as a product $F(s) = \prod_p F_p(s)$, where $\log F_p(s) = \sum_{k=1}^{\infty} \frac{b_{p^k}}{p^{ks}}$ with $|b_{p^k}| \leq Cp^{k\theta}$ for some $\theta < 1/2$ and some constant $C > 0$.

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The series $L(s, \Delta) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^{s+11/2}}$ satisfies (P1), (P2) (with $m = 0$), (P3) with $G(s) = \Gamma(s + 11/2)$ and also (P4)

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Problem: Very few unitary automorphic L-functions are known to satisfy the condition with $\mu_j \geq 0$. Showing this, is equivalent to showing the Generalised Ramanujan Conjectures at infinity.

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Advantage: All generic unitary automorphic L-functions (of GL_n) are known to be in $\mathcal{G}$. Moreover, the $1/2$ that appears above is the analogue of the $1/2$ that appears in (P4). This is the Jacquet-Shalika bound.

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More: The theorems will also be valid for series that arise as factors of unitary automorphic L-functons.

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If $F(s)$ lies in $\mathcal{G}^{\#}$ and satisfies two different functional equations for factors $G_1(s)$ and $G_2(s)$, then $G_1(s) = cG_2(s)$ for some $c \in \mathbb{C}$.

The proofs of the above statements are variations on the arguments made in $\mathcal{S}$ and $\mathcal{S}^{\#}$ by Conrey-Ghosh (C-G) and Kaczorowski-Perelli (K-P)

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The proof of the theorem above, is an easy modification of arguments of Richert and C-G made for $\mathcal{S}$.

Factorisation

Notice that the class $\mathcal{L}^{\#}$ forms a monoid under multiplication of series. An element F in $\mathcal{L}^{\#}$ is said to be primitive if $F = F_1 F_2$ in $\mathcal{L}^{\#}$ implies that either F_1 or F_2 is a unit.

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Of course, one can conjecture the same for the class $\mathcal{L}$.

The case $d_F = 1$

Theorem

Let F be in $\mathcal{L}^{\#}$ and $d_F = 1$. Then (upto factors of degree 0), $F(s)$ is a linear combination of L-functions of the form $L(s + it, \chi)$.

This theorem was proved by K-P for $\mathcal{S}^{\#}$ (Acta Mathematica, 1999). Soundararajan gave a very short and elementary proof in 2005. I am able to modify Soundararajan's proof to get the result for $\mathcal{L}^{\#}$.

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Again, the above result has been known for $\mathcal{S}$ by the work of (K-P). But all the functions $L(s,\pi)$ are not known to lie in $\mathcal{S}$. They are known to lie in $\mathcal{L}$ though!

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The proof here involves a much more serious departure from existing arguments. In particular we need to use the celebrated results of Montgomery and Montgomery-Vaughan on L^2 norms.

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We should remark that our method allows for a considerable shortening of the difficult proof of the same result of K-P in $\mathcal{S}^{\#}$ (Inventiones, 2002). However, they have gone further and proved (Annals, 2012)

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We should remark that our method allows for a considerable shortening of the difficult proof of the same result of K-P in $\mathcal{S}^{\#}$ (Inventiones, 2002). However, they have gone further and proved (Annals, 2012)

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If $1 < d < 2$, $\mathcal{S}_d^{\#} = \emptyset$.

We believe that our techniques should be good enough to 1) extend their results to $\mathcal{L}^{\#}$ and 2) yield a much shorter and different proof.

Coda

Using the theorem above and arguments similar to those of K-P, we can prove

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This is work in progress.